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Hybrid Representations of Coupled Nonparametric and Parametric Models for Dynamic Systems

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Parametric modeling of stochastic systems has proven useful for systems with well-defined and well-structured sources of uncertainty. The suitability of such models is usually indicated by small levels of uncertainty associated with their parameters. The parametric model may not be efficiently employed to deal with problems associated with a high level of uncertainty, particularly due to the modeling uncertainty. The class of so-called nonparametric stochastic models has recently been introduced to address this specific issue and found to be useful to some extent. This paper presents a coupling technique adapted to the receptance frequency-response-function matrix that will be useful for analyzing a complex dynamic system, particularly when it consists of several stochastic subsystems, each of which is individually deemed to be suitable for either a parametric model or a nonparametric model. Such a complex dynamic system is otherwise difficult to analyze. The existing nonparametric approach was, to date, applied to the real-valued positive-definite/semidefinite random system matrix: for example, mass, damping, and stiffness matrices. In the present work, the nonparametric approach is also employed with the complex-valued symmetric receptance frequency-response-function matrix, now acting as the system matrix, by having recourse to Takagi's factorization.

I. Introduction

TWO types of uncertainty are of particular interest in connection with the dynamic systems: *modeling uncertainty* and *data uncertainty*. Modeling uncertainty can be further decomposed into *mechanical uncertainty* and *probabilistic uncertainty*. Mechanical uncertainty results from several simplifying assumptions invoked while developing a physics-based mathematical model of the physical phenomenon at hand. Probabilistic uncertainty, on the other hand, stems from the introduction of probabilistic assumptions about model parameters such as geometry, boundary conditions, and constitutive parameters. A few representative examples in which mechanical uncertainty is present include mechanical models of joints, one- and two-dimensional beams, and plate models. Even the 3-dimensional theory of elasticity encompasses several assumptions introduced for mathematical convenience that do not satisfy the true behavior of the system being analyzed. Assumptions of statistical independence between two random system parameters and the assignment of a particular probability distribution law to various system parameters are, for their part, sources of probabilistic uncertainty. Data uncertainty, on the other hand, is characterized by the uncertainty associated with the data collected from experimental measurements for estimation of the statistical and probabilistic features of the model parameters. *Paucity of data* and *experimental uncertainty* due, for instance, to human error or imperfect experimental conditions are typical sources of data uncertainty. Note that data interpretation using some physics-based model is an integral part of any experimental effort. Although this seems to challenge the preceding delineation of uncertainties, the matter can be readily remedied by recognizing that the interpretation of experimental data typically relies on models different from those used for system-level prediction. Moreover, these models usually exhibit less uncertainty than their system-level counterparts.

Often, models of certain parts of a complex system are more accurate than those in other parts; for instance, the mechanical model of a part constituted of a slender-beam structure is generally better than a simplified mechanical model of a complex joint. This indicates that the uncertainties resulting from the mechanical model are not homogeneous throughout the system. Uncertainties resulting from probabilistic uncertainty are also not spatially homogeneous, because some parameters of a certain part of a complex system might be truly statistically independent, whereas the assumption of statistical independence for other parameters may be invoked as a mere mathematical convenience. Uncertainties resulting from the data uncertainty are again not spatially homogeneous, because the data may not be uniformly available throughout the complex system. It is therefore clear that, in general, uncertainties in a complex system are spatially nonhomogeneous.

The analysis of a complex dynamic system with such nonhomogeneous uncertainties is typically quite involved, both to set up and to resolve numerically. It may not even be possible to analyze the built-up structure because of the presence of the spatially nonhomogeneous uncertainties. To model all of the uncertainties in a complex dynamic system and to solve the global stochastic equations, it is useful and convenient to decompose it into several smaller subsystems such that uncertainty in each subsystem is spatially homogeneous. Each of these subsystems can be analyzed separately using methods that are adapted for it, with the system response being synthesized suitably from these subsystem analyses. A subsystem having a lower level of modeling and data uncertainty or having a relatively fewer number of random system parameters can be analyzed by using the parametric approach requiring the knowledge of local system parameters. On the other hand, a subsystem having a higher level of uncertainty due to modeling uncertainty and data uncertainty or having a large number of random system parameters can be analyzed by using the recently proposed approach called the nonparametric approach [1–5], which can bypass the concept of local system parameters and directly model the uncertainty in system-level matrices. The present work is motivated by the observation that, often, the behavior in certain parts of a complex system is well captured by parametric models associated with partial differential equations and conservation laws. Suitably providing for this knowledge can surely reduce the overall uncertainty about system behavior.

This paper presents a methodology that permits the coupling between subsystems analyzed using a mixture of parametric and nonparametric formalisms. The approach is based on the pointwise enforcement of dynamic equilibrium across the sampling space. Following a review of nonparametric models, the proposed hybrid

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approach is described in detail. The method is then demonstrated on a numerical example of a built-up system with joints.

II. Nonparametric Model

In this section, an overview is provided of the nonparametric approach [1,2] to model uncertainties in dynamic systems. This model differs from the parametric modeling of uncertainties in that it does not require information on the local parameters of the system being analyzed. The nonparametric framework allows one to construct a probability density function (pdf) of the random system matrices directly based on partial knowledge of the system. It is directed toward constructing the probability models of the random system matrices without consideration of the parameterization of the dynamic model of the system. One thus considers the problem of constructing the probability space directly on the finite-dimensional representation of the dynamic system: namely, $(\mathbb{M}_n^S(\mathbb{R}), \mathcal{F}, P_{\mathbf{A}})$, where $\mathbb{M}_n^S(\mathbb{R})$ is the set of all real $n \times n$ symmetric matrices, $\mathcal{F}$ is the σ algebra of subsets of $\mathbb{M}_n^S(\mathbb{R})$, $P_{\mathbf{A}}$ is the probability measure on $\mathcal{F}$, and $\mathbf{A}$ represents the associated random matrix variate. It is assumed here that $P_{\mathbf{A}}$ admits a pdf

$$p_{\mathbf{A}}\colon \mathbb{M}_n^S(\mathbb{R}) \mapsto \mathbb{R}^+ = [0, \infty)$$

with $\text{supp}(p_{\mathbf{A}}) = \mathbb{M}_n^+(\mathbb{R}) \subset \mathbb{M}_n^S$ being the set of symmetric positive-definite real matrices, such that

$$dP_{\mathbf{A}}(A) = p_{\mathbf{A}}(A)\tilde{d}A$$

where $\tilde{d}A$ can be interpreted as a volume element in $\mathbb{M}_n^S(\mathbb{R})$ and is defined to be the wedge product or the exterior product [6] of the independent elements of the differential form of the matrix dA, the (i, j)th element of dA is simply defined [7] as dA_{ij} where A_{ij} is the (i, j)th element of A. There are $n(n+1)/2$ such independent elements. In the context of random matrix theory [7,8], for symmetric matrices, $\tilde{d}A$ is then given by $\bigwedge_{1\le i\le j\le n} dA_{ij}$, where $\bigwedge$ indicates the wedge product. In the present context, the wedge product reverts to be the natural volume element

$$\prod_{1\le i\le j\le n} dA_{ij}$$

in the Euclidean space $\mathbb{R}^{n(n+1)/2}$ that is topologically equivalent to $\mathbb{M}_n^S(\mathbb{R})$. The nonparametric approach [1,2], in the context of stochastic mechanics applications is, however, developed by considering an Euclidean structure on $\mathbb{M}_n^S(\mathbb{R})$ to define the following volume element:

$$\tilde{d}A = 2^{n(n-1)/4} \prod_{1\le i\le j\le n} dA_{ij}$$

which differs by a multiplicative factor $2^{n(n-1)/4}$ from the natural volume element. Note that it is, however, feasible to reformulate the theory behind the nonparametric approach by using the natural volume element

$$\prod_{1\le i\le j\le n} dA_{ij}$$

with minor changes ([9], Chapter 5).

In constructing the probability measure, the principle of maximum entropy [10], as initially introduced by Jaynes [11,12] for discrete random variables, has been used [1]. The maximum-entropy (MaxEnt) principle yields a constrained optimization problem with the objective being to maximize Shannon's [13,14] measure of entropy subject to constraints that represent available information. These constraints are typically in the form of observed statistics (mean, variance, etc.) of the random variate. In the present case, entropy can be interpreted as a measure of relative uncertainty [10] associated with the probability distribution of the random matrix variate. This uncertainty is not about which realization of the random matrix variate will be observed, but it represents the scatter in the probability distribution of the random matrix variate. The basic idea of the MaxEnt principle is to choose the probability distribution function (PDF) (contrast it with pdf abbreviated earlier for probability density function) with maximum uncertainty out of all of the PDFs that are consistent with the given set of constraints. Any other PDF would be associated with unwarranted assumptions about the systems that are not supported by the available information [11,12].

Note that according to the usual statement of the MaxEnt principle, the uniform distribution is assumed to represent a state of maximum uncertainty. Interestingly, the pdf resulting from the application of Jaynes's [11,12] MaxEnt principle on the continuous pdf ([10], Sec. 2.5.2) is the same as the pdf resulting from the use of the of minimum directed divergence (minimum cross entropy) [15,16], provided the prior PDF is uniform [17–19]. Under this uniform prior PDF, the principle of minimum directed divergence chooses, out of all probability distributions satisfying the given constraints, the one that is closest to the uniform distribution. Geometrically, it corresponds to the pdf, in a space consisting of pdfs, that has the minimum *directed* distance computed by using the cross-entropy measure [15,16], with respect to the point in this space representing the uniform distribution. Note though that the space of probability distributions equipped with *directed* distance measured in terms of Kullback–Leibler's [15] cross entropy is not a metric space.

Let us denote the mass matrix of the mean dynamic system by $\underline{M} \in \mathbb{M}_n^+(\mathbb{R})$ and its Cholesky decomposition ([20], Theorem 14.5.11) by

$$\underline{M} = L_M^T L_M \tag{1}$$

where L_M is an upper triangular matrix in the set $\mathbb{M}_n(\mathbb{R})$ of all real matrices of size $n \times n$, where n is the total degrees of freedom (DOF) of the mean finite element model, and T represents the transpose operator. The mean stiffness matrix $\underline{K}$ is positive definite for a fixed structure and positive semidefinite for a free structure and consequently has the following Cholesky decomposition ([20], Theorem 14.5.16):

$$\underline{K} = S_K^T S_K$$

where S_K is an upper triangular matrix in $\mathbb{M}_n(\mathbb{R})$ with a fixed structure and is an almost upper triangular matrix in $\mathbb{M}_{m,n}(\mathbb{R})$ for free structure with $(n - m) \le 6$ being the number of rigid-body modes of the system, where $\mathbb{M}_{m,n}(\mathbb{R})$ is the set of all real matrices of size $m \times n$. The random system mass matrix $\mathbf{M}$ and random system stiffness matrix $\mathbf{K}$ are then written as [21]

$$\mathbf{M} = L_M^T \mathbf{G}_M L_M \quad \text{and} \quad \mathbf{K} = S_K^T \mathbf{G}_K S_K$$

where $\mathbf{G}_M$ and $\mathbf{G}_K$ are second-order [see the end of this section] random matrix variates, respectively, in $\mathbb{M}_n^+(\mathbb{R})$ and $\mathbb{M}_m^+(\mathbb{R})$ with $E\{\mathbf{G}_M\} = I_n$ and $E\{\mathbf{G}_K\} = I_m$, where E is the mathematical expectation operator and I_p is an identity matrix of size $p \times p$ $(p = n, m)$. A similar decomposition exists for the random damping matrix.

Next, a procedure must be developed to sample from the ensemble of random matrices $\mathbf{G}_M$ and $\mathbf{G}_K$ to simulate the realizations of $\mathbf{M}$ and $\mathbf{K}$. This entails developing expressions for the pdf of the random matrices, together with a procedure for sampling from this pdf. Let us generically denote the random matrix to be generated ($\mathbf{G}_M, \mathbf{G}_K$ or the corresponding matrix for damping) by $\mathbf{A} \in \mathbf{M}_n^+(\mathbb{R})$. The pdf $p_{\mathbf{A}}$ is then determined by using the principle of maximum entropy. The nonparametric approach [1,2] only assumes the information of the ensemble means of the system matrices (mass, stiffness, and damping matrices) to be known a priori. These ensemble means can be taken as the system matrices obtained by discretizing a nominal continuous system in view of analyzing it using the finite element model (FEM). Subsequent uses of the following three constraints yields the MaxEnt pdf $p_{\mathbf{A}}$:

1) The total area under $p_{\mathbf{A}}(\cdot)$ over $\mathbf{M}_n^+(\mathbb{R})$ must be unity (normalization constraint).

2) The ensemble mean matrix $\underline{A}$ is known and assumed to be given by the matrix corresponding to a nominal system available on hand (ensemble mean constraint).

3) The moments of the response random variables must exist and this existence is related to the existence of the moments of the random system matrices (existence of the moments of the response constraint).

Here, it can be shown that the last constraint implies that ([2], page 1985) for mass, damping, and stiffness matrices,

$$E\{\|\mathbf{A}^{-1}\|_F^{\gamma}\} < \infty$$

where $\gamma \geq 1$ is a positive integer and $\|\cdot\|_F$ is the Frobenius norm defined by

$$\|A\|_F = \langle A, A^T\rangle^{1/2} \equiv \{\mathrm{tr}(AA^T)\}^{1/2} = \left(\sum_{ij}|a_{ij}|^2\right)^{1/2}$$

where a_{ij} is the (i,j)th element of A.

The pdf $p_{\mathbf{A}}$ thus obtained by maximizing the entropy subject to the preceding three constraints is characterized by three parameters: $\underline{A}$, λ, and n. Here, $1-\lambda$ is one of the Lagrange multipliers, with $\lambda > 0$, associated with the last constraint. Because $E\{\mathbf{G}_M\}$ and $E\{\mathbf{G}_K\}$ and the matrix corresponding to the random damping matrix are identity matrices, we consider the case when $\underline{A}$ is an identity matrix. Then λ is given by [2]

$$\lambda = \frac{(1-\delta_A^2)}{2\delta_A^2}n + \frac{1+\delta_A^2}{2\delta_A^2} \tag{2}$$

where $\delta_A > 0$ is the dispersion parameter indicating a scalar-valued measure of the level of scatter associated with the corresponding system matrix and defined by

$$\delta_A = \left\{\frac{E\{\|\mathbf{A}-\underline{A}\|_F^2\}}{\|\underline{A}\|_F^2}\right\}^{1/2} \tag{3}$$

From the convergence study of the second-order moment of the inverse random matrix as $n \to \infty$, δ_A must satisfy the following relation [2]:

$$0 < \delta_A < \sqrt{\frac{n_0+1}{n_0+5}}, \quad \forall\ n \geq n_0 \tag{4}$$

where $n_0 \geq 1$ is a fixed integer. This condition then guarantees the existence of the mean and the second-order moment of $\mathbf{A}^{-1}$ (as required for the random matrices associated with mass, damping, and stiffness). The upper bound of δ_A in Eq. (4) is a monotonically and strictly increasing function of n_0 with

$$\min\sqrt{(n_0+1)/(n_0+5)} = 0.5774$$

for $n_0 = 1$ and

$$\sup\sqrt{(n_0+1)/(n_0+5)} = 1$$

Because n and δ_A are known for a given problem, n_0 can be chosen from $\{1,2,\ldots,n\}$ such that the specified δ_A satisfies Eq. (4). In the context of practical problems, $n \geq 1$, and δ_A typically satisfies the relation expressed by Eq. (4). It then implies that $\lambda \gg 1$ by Eq. (2).

The PDF associated with the pdf estimated using the preceding MaxEnt formulation is the Wishart distribution $W(m_A, (1/m_A)\underline{A})$, where $m_A = n-1+2\lambda = (n+1)/\delta_A^2$ [22–24]. The usual Wishart distribution is only defined when m_A is an integer and $m_A \geq n$. However, in the current random matrix literature [22,23], the term Wishart distribution is extended to cover the case when m_A is not an integer and $m_A > n-1$. In the latter case, this distribution is also referred to as matrix-variate gamma distribution [23,25] with slightly different parametrization. In the present work and for conciseness, no distinction will be made between the usual Wishart distribution and the matrix-variate gamma distribution.

The procedures for generations of realizations of the random matrix $\mathbf{A}$ are discussed next.

A. Monte Carlo Simulation of A

For many applications, n is sufficiently large, and in such cases, m_A can be reasonably approximated as an integer. Then the following simple form [24] of the random matrix $\mathbf{A}$ can be used to sample $\mathbf{A} \sim W(m_A, (1/m_A)\underline{A})$ (the symbol $\sim$ should be interpreted as *distributed as*):

$$\mathbf{A} = \frac{1}{m_A}\sum_{j=1}^{m_A}(L_A^T\mathbf{U}_j)(L_A^T\mathbf{U}_j)^T \tag{5}$$

where L_A is defined by the Cholesky decomposition of $\underline{A}$ (i.e., $\underline{A} = L_A^T L_A$), and $\mathbf{U}_j$ are independent and identically distributed $\mathbb{R}^n$-valued normal random vectors [i.e., $N(\mathbf{0}, I_n)$]. This form requires sampling statistically independent $m_A n$ standard normal random variables to produce $\mathbf{A}$ consisting of only $n(n+1)/2$ distinct random variables $\mathbf{A}_{ij}$, where $\mathbf{A}_{ij}$ is the (i,j)th element of $\mathbf{A}$. A more computationally efficient method based on Barlett's decomposition [26] has been proposed that requires only $n(n-1)/2$ statistically independent normal random variables and n statistically independent chi-square random variables [27]. This is summarized in Algorithm 1.

In Algorithm 1, $\chi^2_{m_A-i+1}$ represents chi-square distribution with $(m_A - i + 1)$ degrees of freedom. The computational benefit of the preceding algorithm, when compared with the brute-force Monte Carlo simulation (MCS) of $\mathbf{A}$ based on Eq. (5), can be realized, particularly for large n.

As already indicated, Eq. (5) or the Algorithm 1 is valid only when m_A can be rounded off to the nearest integer without significant modeling error. However, this introduces probabilistic uncertainty in the model. To avoid introducing this probabilistic uncertainty, albeit small, the exact simulation technique can be used if m_A is not an integer. When $m_A (> n-1)$ is not an integer, the previous algorithm can be readily adapted to sample $\mathbf{A}$ by reinterpreting the chi-square distribution $\chi^2_{m_A-i+1}$ in step 2 as the gamma distribution $G((m_A - i + 1)/2, 1/2)$ [9], where the pdf associated with the gamma distribution $G(k,\gamma)$ is given by

$$p(t) = \frac{\gamma^k}{\Gamma(k)}t^{k-1}\exp(-\gamma t), \qquad k,\gamma > 0, \qquad t > 0$$

Sampling procedures for this case have been recently proposed [1,2]. Further insights and relevant theoretical details that help to prescribe the preceding simulation procedures for $\mathbf{A}$ are available elsewhere [1,2,9,22,23].

B. Related Works and Further Remarks

Note that the preceding nonparametric formulation has been developed along similar lines to those used in constructing a probability space for the Gaussian orthogonal ensemble (GOE) [8]. However, unlike the statistically independent entries in GOE matrices, the entries $\mathbf{A}_{ij}$ of the random matrix $\mathbf{A} \in \mathbb{M}_n^+(\mathbb{R})$ are not statistically independent. A comparative study using statistical samples of random matrices computed, respectively, according to the nonparametric approach and the GOE has been conducted to show the superiority of the nonparametric approach over the GOE

Algorithm 1 Sampling from Wishart distribution, $W(m_A, (1/m_A)\underline{A})$

Input: dimension of matrix: n, and parameter: $m_A \geq n$.
Output: samples of $\mathbf{A} \sim W(m_A, (1/m_A)\underline{A})$.
Step 0: consider a lower triangular matrix $\mathbf{T} = (\mathbf{t}_{ij})$.
Step 1: generate statistically independent $\mathbf{t}_{ij} \sim N(0,1)$ for $1 \leq j < i \leq n$.
Step 2: generate statistically independent $\mathbf{y}_i \sim \chi^2_{m_A-i+1}$ for $i = 1,\ldots,n$.
Take $\mathbf{t}_{ii} = \sqrt{\mathbf{y}_i}$.
Step 3: $\mathbf{A} = (1/m_A)L_A^T(\mathbf{T}\mathbf{T}^T)L_A \sim W(m_A, (1/m_A)\underline{A})$.

approach in the context of structural dynamics problems [28]. Comparisons have also been conducted between nonparametric and parametric models to highlight features of the former in the low-frequency regime. The nonparametric approach has been applied to both frequency-domain [1,28] and time-domain [2,29] problems in structural dynamics.

This section is concluded by noting that given partial information separately for each of the system matrices with no information regarding the statistical dependency among these system matrices, the MaxEnt principle also implies that these system matrices are statistically independent of each other. The statistical independence of mass, damping, and stiffness matrices is consistent with assumptions, which are difficult to describe and quantify, regarding the sources of uncertainty, such as the manufacturing process or microstructure morphology. In Sec. III, we present a methodology that relaxes this assumption of independence by constructing a probabilistic model directly on the frequency-response-function (FRF) matrices. In this case, we introduce the alternative assumption of statistical independence between the values of the FRF at different frequencies.

III. Nonparametric Model for a Complex FRF Matrix

The receptance FRF matrix $H(\omega)$ is expressed by

$$H(\omega) = (K - \omega^2 M + \iota(\omega C + D))^{-1}$$

where K, M, C, and D are, respectively, the system matrices of stiffness, mass, viscous damping, and structural damping; ω is the forcing frequency; $\iota = \sqrt{-1}$; and $H(\omega)$ is a complex symmetric matrix and consequently has the following Takagi factorization ([30], Corollary 4.4.4):

$$H(\omega) = U(\omega)\Sigma(\omega)U(\omega)^T \tag{6}$$

where $U(\omega) \in \mathbb{U}(n)$ is a unitary matrix, with $\mathbb{U}(n)$ being the unitary group of unitary matrices of size $n \times n$ ([30], page 69). For the sake of simplification of notation, the dependence on ω will be suppressed in some part of the ensuing discussions if deemed not important. The set of orthonormal eigenvectors of $H(\omega)H(\omega)^*$ constitutes the columns of $U(\omega)$, and the positive square roots of the corresponding eigenvalues of $H(\omega)H(\omega)^*$ are the corresponding diagonal entries of $\Sigma(\omega)$. Here, * represents the elementwise conjugate operator.

Takagi's factorization for a complex symmetric matrix is a special case of singular-value decomposition (SVD) for a symmetric matrix. The SVD exists for any matrix $A \in \mathbb{M}_{m,n}(\mathbb{C})$ [where $\mathbb{M}_{m,n}(\mathbb{C})$ is the set of all complex matrices of size $m \times n$] such that $A = U\Sigma W^\dagger$, with $U \in \mathbb{U}(m)$ and $W \in \mathbb{U}(n)$ being unitary matrices and the diagonal entries of the diagonal matrix Σ being the nonnegative square roots of the eigenvalues of $AA^\dagger$ (where $\dagger$ represents the conjugate-transpose operator). In Takagi's factorization for a complex symmetric matrix, it turns out that $U = W^*$. Now, at any fixed ω, define $A = H(\omega)H(\omega)^*$. As $H(\omega)$ is symmetric, $H(\omega)^* = H(\omega)^\dagger$. Consequently, $A = A^\dagger$ is Hermitian and also positive definite because

$$x^\dagger Ax = x^\dagger H(\omega)H(\omega)^* x = (H(\omega)^\dagger x)^\dagger (H(\omega)^\dagger x)$$
$$= \|H(\omega)^\dagger x\|_2^2 > 0 \quad \forall \text{ nonzero } x \in \mathbb{C}_n$$

where $\|\cdot\|_2$ is a Hermitian or Euclidean norm ([30], pp. 264) on $\mathbb{C}^n$. Therefore, all of the eigenvalues of $H(\omega)H(\omega)^*$ are positive. If we denote the group of all of the diagonal matrices of size $n \times n$ with positive diagonal entries by $\mathbb{D}_n^+$, then $\Sigma \in \mathbb{D}_n^+ \subset \mathbb{M}_n^+(\mathbb{R})$. Note that Eq. (6) can also be written as $H(\omega) = V^T V$, where $V = (U\Sigma^{1/2})^T$ with

$$\Sigma^{1/2} = \text{diag}(+\sqrt{\sigma_1}, \ldots, +\sqrt{\sigma_n})$$

where σ_j is the jth diagonal element of Σ. Hence, the receptance FRF matrix of the mean dynamic system $\underline{H}(\omega)$ has the following decomposition:

$$\underline{H}(\omega) = V_{H(\omega)}^T V_{H(\omega)}$$

which can be compared with Eq. (1). Now the random receptance FRF matrix $\mathbf{H}(\omega)$ can be written as

$$\mathbf{H}(\omega) = V_{H(\omega)}^T \mathbf{G}_{H(\omega)} V_{H(\omega)} \tag{7}$$

where $\mathbf{G}_{H(\omega)}$ is the random matrix variate (with all of its moments being finite) in $\mathbb{M}_n^+(\mathbb{R})$ and $E\{\mathbf{G}_{H(\omega)}\} = I_n$. The last constraint condition on the existence of moments of response, as mentioned earlier, also guarantees the existence of the moments of $\mathbf{A}$, implying ([2], p. 1985) for the receptance FRF matrix,

$$E\{\|\mathbf{A}\|_F^\gamma\} < \infty$$

as required for the random matrix associated with the receptance FRF matrix to enforce the condition for the existence of moments of random response.

A probability model for $\mathbf{G}_{H(\omega)}$ can then be developed in exactly the same way as described earlier for $\mathbf{G}_M$ and $\mathbf{G}_K$ in Sec. II. Simulation of $\mathbf{G}_{H(\omega)}$, and therefore of $\mathbf{H}(\omega)$, follows from the constructed probability model of $\mathbf{G}_{H(\omega)}$ by virtue of Eq. (7).

Note that the assumption of statistical independence among the random mass, damping, and stiffness matrices that prevails in the existing nonparametric formulation is relaxed in the proposed FRF-based nonparametric formulation. This is consistent with treating the FRF matrix as the coarsest-level description of the dynamic system, rather than the standard description parameterized by mass, damping, and stiffness matrices. As mentioned previously, however, the present approach has shifted the assumption of statistical independence from a material context relating to microstructure and manufacturing process to a dynamics context. Specifically, FRF matrices at different frequencies are assumed to be realizations from statistically independent random matrices. These are characterized independently, resulting in a corresponding growth in the computational burden.

The proposed FRF-based nonparametric formulation is best adapted to cases in which response statistics of a dynamic system are sought as functions of ω. In these cases, the FRF matrix at a given frequency reflects the behavior of a consistent dynamic system for which the dynamic properties are not assumed to be statistically independent. Also note that a matrix-valued stochastic process evolving over ω can be constructed for $\mathbf{G}_{H(\omega)}$ within the present FRF-based nonparametric formulation by adapting existing approaches of constructing the matrix-valued stochastic process [31] at the expense of further experimental and computational cost. It is finally noted in this context that the higher the frequency range of interest (see [32–34] for related theories and applications), the more applicable the assumption of independent FRF matrices at different frequencies.

Furthermore, the proposed FRF-based nonparametric formulation is more suitable from practical considerations, as experimental measurements are usually *directly* available on the FRF matrix and not on the mass, damping, or stiffness matrices. Natural estimators or optimization-based techniques have been proposed to determine the mean and dispersion parameters in the probability models of the random mass, damping, and stiffness matrices when the existing nonparametric formulation is employed [4,35,36]. Similar techniques can also be readily adapted to estimate the parameters of the probability model of the random FRF system matrix $\mathbf{H}(\omega)$. In this case, the parameters, in general, will be evaluated independently for each ω.

IV. Coupling Nonparametric Model and Parametric Model

The coupling technique used in the current work to combine the nonparametric model and the parametric model is based on the receptance FRF matrices of the uncoupled subsystems. Several such FRF-based coupling techniques exist in the literature [37–41]. One such method has already been used by the authors in the context of a

different class of vibration problem [42] and will be used in the current work. A description of this coupling technique follows next.

Consider the mean built-up system consisting of several subsystems. Also consider one constituent subsystem. Denote this subsystem by j and the complex FRF matrix of the subsystem by $\underline{H}^{(j)}(\omega)$ when it is isolated from the other adjoining subsystems of the built-up structure. The procedure of construction of the matrix $\underline{H}^{(j)}(\omega)$ is not relevant in the coupling technique described here. The receptance FRF matrix can be constructed by using any available technique: for example, modal analysis, direct inversion of the dynamic stiffness matrix of the corresponding uncoupled finite element model, or even by experimentally identifying this FRF matrix. Note that the subsystem j could be either classically or nonclassically damped and viscously or hysteretically damped. We need to adopt an appropriate method to analyze the subsystem to compute $\underline{H}^{(j)}(\omega)$. Given $\underline{H}^{(j)}(\omega)$, the response of the subsystem in frequency domain can be obtained from the following equation:

$$\underline{X}^{(j)}(\omega) = \underline{H}^{(j)}(\omega)\underline{F}^{(j)}(\omega) \tag{8}$$

where

$$\underline{X}^{(j)}(\omega) = [\,\underline{X}_1^{(j)} \quad \underline{X}_2^{(j)} \quad \cdots \quad \underline{X}_{n_j}^{(j)}\,]^T \in \mathbb{C}^{n_j}$$

is the response of subsystem j, where n_j is the total number of DOF (displacements or/and rotations) of the subsystem j. Similarly,

$$\underline{F}^{(j)}(\omega) = [\,\underline{F}_1^{(j)} \quad \underline{F}_2^{(j)} \quad \cdots \quad \underline{F}_{n_j}^{(j)}\,]^T \in \mathbb{C}^{n_j}$$

is the vector of forcing components (including the coupling forces and the externally applied forces) at the n_j DOF, when the subsystem j is isolated from the other adjoining subsystems. Then $\underline{H}^{(j)}(\omega)$ is an $n_j \times n_j$ complex symmetric matrix. The (s,t)th element of this matrix, $\underline{H}_{s,t}^{(j)}(\omega)$, represents the response (displacement/rotation) of the subsystem j at the frequency ω at the sth DOF due to a unit force (load/moment) acting at the tth DOF.

Let all of the excitation points on subsystem j be denoted by $(\hat{I},\ldots,\hat{Z})$ and all of the coupling points by $(I,\ldots,Z)$. Consider one of these coupling points as being denoted by k $[k \in (I\cdots Z)]$. Also suppose that there are N_k number of subsystems denoted by $(j,l,\ldots,s)$ connected at this coupling point k. In addition, consider that at this coupling point, there are a total p_k number of DOF ($p_k \le 6$) that must maintain the continuity of the corresponding responses among the subsystems $(j,l,\ldots,s)$, meeting at the coupling point. For subsystem j, let us denote these DOF that would generate either a coupling force or a coupling moment at the coupling point k, when the subsystem j would be isolated from the other adjoining subsystems, by $m_j(o_k)$ $(o_k = 1,2\cdots p_k)$. The response at the DOF $m_j(o_k)$ $(o_k = 1,2\cdots p_k)$ then can be expressed by

$$\underline{X}_{m_j(o_k)}^{(j)}(\omega) = \sum_{r=1}^{n_j} \underline{H}_{m_j(o_k),r}^{(j)}(\omega)\underline{F}_r^{(j)}(\omega) \tag{9}$$

Note that the force components $\underline{F}_r^{(j)}(\omega)$ contain both the known externally applied forces and the unknown coupling forces resulting from the isolation of the subsystem j from the other adjoining subsystems $(l,\ldots,s)$. Hence, it is useful to decompose the right-hand side (RHS) of Eq. (9) into two parts: one containing the contributions from the known components of the externally applied forces and the other containing the contributions from the unknown coupling forces. This is done next.

Let S_j be the set containing the DOF associated with the unknown coupling force components for subsystem j:

$$S_j = \{m_j(o_k),\ o_k = 1,2,\ldots,p_k,\ k \in (I,\ldots,Z)\}$$

If we denote the total number of DOF [at an excitation point q, where $q \in (\hat{I},\ldots,\hat{Z})$] that are associated with the nonzero externally applied known force components by p_q $(p_q \le 6)$ and the corresponding DOF by $m_j(o_q)$ $(o_q = 1,2,\ldots,p_q)$, we have

$$\hat{S}_j = \{m_j(o_q),\ o_q = 1,2,\ldots,p_q,\ q \in (\hat{I},\ldots,\hat{Z})\}$$

As there are a total N_k number of subsystems $(j,l,\ldots,s)$ connected at the coupling point k, one can write a total N_k number of expressions of response for each DOF at this point. During the process of assembling the subsystems, we now need to merge the appropriate DOF of their coupling points. For convenience, we sort the elements of the sets of the DOF of the subsystems $(j,l,\ldots,s)$ connected at the coupling point k, $\{m_j(o_k),\ o_k = 1,2,\ldots,p_k\}$, $\{m_l(o_k),\ o_k = 1,2,\ldots,p_k\},\ldots,\{m_s(o_k),\ o_k = 1,2,\ldots,p_k\}$ such that the o_kth element of each of the sets,

$$m_i(o_k) \qquad i = j,l\cdots s \quad \forall\ o_k = 1,2\cdots p_k$$

refers to the same DOF of the assembled structure. Hence, from the condition of compatibility of the response at the merged DOF of the assembled system, the following $N_k - 1$ number of equations can be formed (after performing some rearrangement-type operations) for each o_k $(o_k = 1,2,\ldots,p_k)$:

$$\begin{aligned}
&\sum_{r\in S_j} \underline{H}_{m_j(o_k),r}^{(j)}(\omega)\underline{F}_r^{(j)}(\omega) - \sum_{r\in S_l} \underline{H}_{m_l(o_k),r}^{(l)}(\omega)\underline{F}_r^{(l)}(\omega)\\
&\quad = -\sum_{\hat{r}\in\hat{S}_j} \underline{H}_{m_j(o_k),\hat{r}}^{(j)}(\omega)\underline{F}_{\hat{r}}^{(j)}(\omega)\\
&\quad + \sum_{\hat{r}\in\hat{S}_l} \underline{H}_{m_l(o_k),\hat{r}}^{(l)}(\omega)\underline{F}_{\hat{r}}^{(l)}(\omega)\ \cdots \text{ first equation}
\end{aligned} \tag{10a}$$

$$\vdots$$

$$\begin{aligned}
&\sum_{r\in S_j} \underline{H}_{m_j(o_k),r}^{(j)}(\omega)\underline{F}_r^{(j)}(\omega) - \sum_{r\in S_s} \underline{H}_{m_s(o_k),r}^{(s)}(\omega)\underline{F}_r^{(s)}(\omega)\\
&\quad = -\sum_{\hat{r}\in\hat{S}_j} \underline{H}_{m_j(o_k),\hat{r}}^{(j)}(\omega)\underline{F}_{\hat{r}}^{(j)}(\omega)\\
&\quad + \sum_{\hat{r}\in\hat{S}_s} \underline{H}_{m_s(o_k),\hat{r}}^{(s)}(\omega)\underline{F}_{\hat{r}}^{(s)}(\omega)\ \cdots\ (N_k-1)\text{th equation}
\end{aligned} \tag{10b}$$

where subsystem l contains $(P,\ldots,T)$ coupling points [with $k \in (P,\ldots,T)$] and $(\hat{P},\ldots,\hat{T})$ exciting points, and subsystem s contains $(S,\ldots,U)$ coupling points [with $k \in (S,\ldots,U)$] and $(\hat{S},\ldots,\hat{U})$ exciting points, so that

$$S_l = \{m_l(o_k),\ o_k = 1,2,\ldots,p_k,\ k \in (P,\ldots,T)\}$$

$$\hat{S}_l = \{m_l(o_q),\ o_q = 1,2,\ldots,p_q,\ q \in (\hat{P},\ldots,\hat{T})\}$$

$$S_s = \{m_s(o_k),\ o_k = 1,2,\ldots,p_k,\ k \in (S,\ldots,U)\}$$

$$\hat{S}_s = \{m_s(o_q),\ o_q = 1,2,\ldots,p_q,\ q \in (\hat{S},\ldots,\hat{U})\}$$

Note that the RHS of this set of equations is completely known because it contains the known external loads acting on the subsystems. The unknown is the coupling force vector denoted by the concatenated column vector

$$\left[\,\underline{F}_{S_j}^{(j)} \quad \underline{F}_{S_l}^{(l)} \quad \cdots \quad \underline{F}_{S_s}^{(s)}\,\right]^T$$

of $(\underline{F}_{S_t}^{(t)})^T$ $[t \in (j,l,\ldots,s)]$, where $(\underline{F}_{S_t}^{(t)})$ is the row vector consisting of the force components $\underline{F}_r^{(t)}$ $(r \in S_t)$. Sets of equations similar to Eq. (10) are further developed in a similar manner for all of the coupling points of the built-up structure.

The next step is to consider the force equilibrium conditions of the coupling forces (loads and moments) of different subsystems at a common DOF. For coupling point k, as the DOF $m_j(o_k)$, $m_l(o_k)$, and $m_s(o_k)$ (for all $o_k = 1,2,\ldots,p_k$) refer to the same DOF of the assembled structure (because we chose to sort them in this fashion),

we have the following force equilibrium condition at the coupling point k:

$$\underline{F}^{(j)}_{m_j(o_k)}(\omega) + \underline{F}^{(l)}_{m_l(o_k)}(\omega) + \cdots + \underline{F}^{(s)}_{m_s(o_k)}(\omega) = 0 \qquad o_k = 1, 2, \ldots, p_k \tag{11}$$

In this manner, it is possible to form the force equilibrium conditions of the coupling forces at all of the coupling points of the assembled structure.

The sets of equations representing the deflection compatibility conditions [see Eq. (10)] at all of the coupling points lead to a total of

$$\sum_{k=1}^{N_c} [N_k - 1] D_k$$

equations. Here, N_c is the total number of coupling points in the assembled system, $N_k \geq 2$ is the number of subsystems coupled at the coupling point k ($k = 1, 2, \ldots, N_c$), and $D_k \leq 6$ is the total number of DOF that would generate either a coupling load or a coupling moment at the coupling point k when one of the subsystems connected at the coupling point would be isolated from the other adjoining subsystems. The sets of equations representing the force equilibrium conditions [see Eq. (11)] at all of the coupling points of the assembled system result in a total of

$$\sum_{k=1}^{N_c} D_k$$

equations. Consequently, there is a total of

$$\sum_{k=1}^{N_c} N_k D_k$$

equations representing the deflection compatibility and the force equilibrium conditions. On the other hand, there is a total of

$$\sum_{k=1}^{N_c} N_k D_k$$

unknown coupling force components denoted by the unknown concatenated coupling force vector:

$$\left[\underline{F}^{(1)}_{S_1} \quad \underline{F}^{(2)}_{S_2} \quad \cdots \quad \underline{F}^{(j)}_{S_j} \quad \underline{F}^{(l)}_{S_l} \quad \cdots \quad \underline{F}^{(s)}_{S_s} \quad \cdots \quad \underline{F}^{(N_s)}_{S_{N_s}} \right]^T$$

where N_s is the total number of subsystems into which the built-up structure is decomposed. This unknown coupling force vector can be readily obtained by solving the preceding equations representing the deflection compatibility and the force equilibrium conditions. Having calculated all of the coupling forces at all of the coupling points, the response of any subsystem j can be readily computed by using Eq. (8):

$$\underline{X}^{(j)}(\omega) = \underline{H}^{(j)}(\omega) \underline{F}^{(j)}(\omega)$$

Though the procedure is described for the mean built-up system, it remains precisely the same if applied to any other realization of the ensemble of the built-up systems. Therefore, the formulation can be applied to each realization of the ensemble of the systems to compute the ensemble of the responses that can be further processed to evaluate the statistics of the response quantities of interests. This procedure results in dynamic equilibrium of the stochastic system being satisfied samplewise.

This coupling technique allows one to treat a complex dynamic structure as being formed of several simple subsystems, each of which could be analyzed individually and independently of others without having recourse to the global mode shapes of the built-up structure. Analysis of each constituent subsystem is performed by using a method that is most adapted to it. Assemblage of all such analyses at the subsystem level results in equations for the built-up structure. This coupling technique requires, as inputs, the subsystems' FRFs and excitations over a given frequency range and produces the output of the built-up system. The basic output of this technique is the displacement field from which other response quantities (i.e., velocity, acceleration, stress, and strain fields) could be readily obtained.

The formulation is exemplified by considering a structure that consists of a set of three free–free Euler–Bernoulli-beam (parametric) subsystems that are discretely coupled by a set of six axially vibrating rod (nonparametric) subsystems.

V. Illustration and Discussion on Results

Consider the built-up structure shown in Fig. 1. Subsystems 2, 3, 4, 6, 7, and 8 are analyzed by using the nonparametric approach and subsystems 1, 5, and 9 are analyzed by using the parametric approach to compute the respective realizations of FRFs of the uncoupled subsystems. These computed FRFs are used to determine the realizations of the response of the built-up structure. The parametric subsystems are modeled as Euler–Bernoulli beams and the mean subsystems of all of the connecting nonparametric subsystems are modeled as axially vibrating rods. The parametric subsystems are analyzed by using the classical dynamic analysis for continuous systems; subsequently, the lowest 10 modes (including the 2 rigid-body modes) of each subsystem have been retained to compute the realizations of the FRFs by using the modal superposition method.

On the other hand, the mass and stiffness matrices of the mean nonparametric subsystems are computed by using commercially available finite element analysis software: namely, ABAQUS. These mean mass and stiffness matrices are then used to construct the MaxEnt pdfs of the random mass and stiffness matrices by following the nonparametric approach, as described earlier. Damping of these subsystems, however, is treated parametrically. In this sense, the nonparametric subsystems have themselves been characterized by both the parametric (with respect to damping) and nonparametric components (with respect to mass and stiffness matrices). (A problem of nonparametric–parametric nature has been reported earlier in the literature [43].) Three modes are retained (including the rigid-body mode) in the FRF computation of nonparametric subsystems. A uniform distribution $U(5 \times 10^{-4}, 1.5 \times 10^{-3})$ with a mean value of 0.001 is assumed for modal damping over all of the modes for all subsystems.

The cross sections of the parametric subsystems are assumed to be circular, with the radius having a uniform distribution $U(0.0248, 0.0253)$ m and mean radius $r = 0.025$ m. The material is assumed to be isotropic and homogeneous, with material density

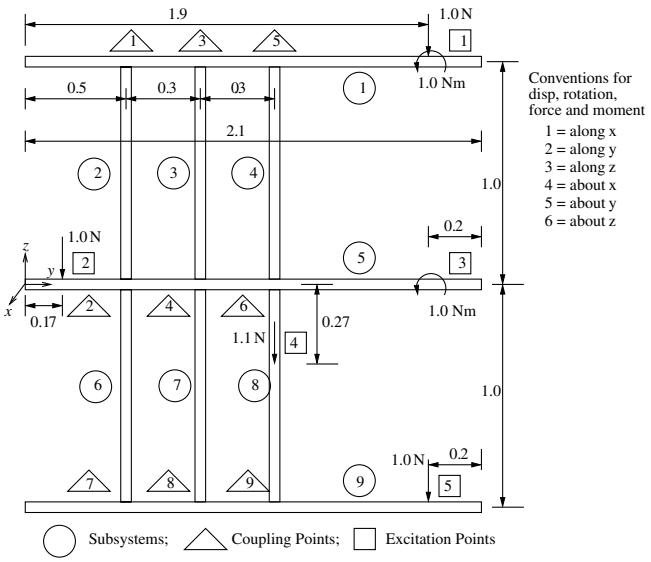

Fig. 1 Mean built-up structure; $E = 2.0 \times 10^{11}$ N/m^2 and $\rho = 7850$ kg/m^3; circular section with radius $r = 0.025$ m; modal critical damping $\xi = 0.001$ over all modes; all dimensions are in meters.

having a uniform distribution $U(7457.5, 8242.5)$ kg/m^3 with mean $\rho = 7850$ kg/m^3 and Young's modulus having a uniform distribution $U(1.9 \times 10^{11}, 2.1 \times 10^{11})$ N/m^2 with mean $E = 2.0 \times 10^{11}$ N/m^2. The length of all of the parametric subsystems is treated as deterministic with a value of 2.1 m. However, the y coordinates of the coupling points and the excitation points (see Fig. 1) are assumed to be random variables having uniform distribution as shown in Tables 1 and 2, where cp_k represents the y coordinate of the coupling point k ($k = 1, \ldots, 9$) and et_k represents the y coordinate of excitation point k ($k = 1, 2, 3, 5$) on parametric subsystems.

The mean subsystems of all of the nonparametric subsystems are assumed to have circular cross sections with radius $r = 0.025$ m, isotropic and homogeneous material with material density $\rho = 7850$ kg/m^3, and Young's modulus $E = 2.0 \times 10^{11}$ N/m^2. The lengths of all of the mean nonparametric subsystems are assumed to be 1.0 m. The dispersion parameters of the associated system matrices are assumed to be $\delta_{K,j} = 0.4$ and $\delta_{M,j} = 0.4$, where $\delta_{M,j}$ and $\delta_{K,j}$ denote, respectively, the dispersion parameters for the mass matrix and stiffness matrix of the nonparametric subsystem j ($j = 2, 3, 4, 6, 7, 8$). This is considered as case 1.

The total number of realizations used in the MCS technique to find the statistics of the response of the built-up structure is 575. This number of realizations of random mass and stiffness matrices and random modal damping parameter (each realization of modal damping parameter remains constant over all modes included in the modal superposition method) are generated to calculate a total of 575 realizations of the receptance FRF matrix at each $\omega = \{1, 2, \ldots, 300\}$ Hz in the frequency band of interest (1–300 Hz) for each uncoupled nonparametric subsystem. Subsequently, these realizations of the FRF matrix $\mathbf{H}^{(j)}(\omega)$ ($j = 2, 3, 4, 6, 7, 8$) are used to estimate the values of the associated dispersion parameters per Eq. (3) by using

$$\delta_{H,j}(\omega) = \left[\frac{1}{m}\left(\sum_{u=1}^{m} \| H^{(j)}(\omega; u) - \underline{H}^{(j)}(\omega)\|_F^2\right) \Big/ \| \underline{H}^{(j)}(\omega)\|_F^2\right]^{1/2} \tag{12}$$

where $m = 575$, and $H^{(j)}(\omega; u)$ ($u = 1, \ldots, m$) is the uth realization of $\mathbf{H}^{(j)}(\omega)$. The results are plotted in Fig. 2. It can be seen that the dispersion parameter of random FRF matrix of each uncoupled nonparametric subsystem remains almost constant over the frequency range of interest (1–300 Hz); consequently, $\delta_{H,2} = 0.2513$, $\delta_{H,3} = 0.2542$, $\delta_{H,4} = 0.2462$, $\delta_{H,6} = 0.2406$, $\delta_{H,7} = 0.2510$, and $\delta_{H,8} = 0.2799$ have been considered in the second phase of analysis (case 2) when realizations of the random

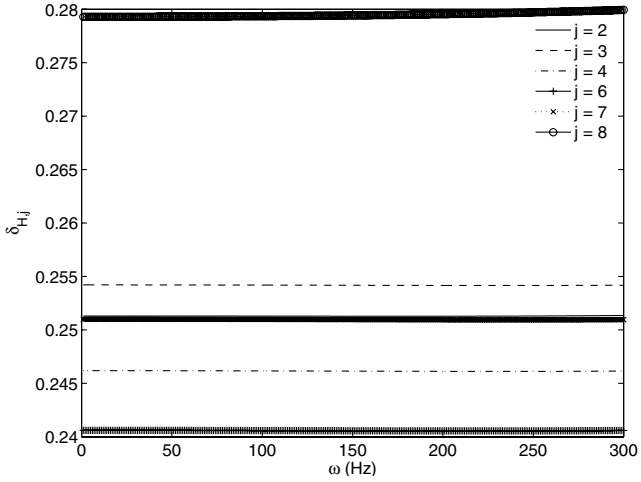

Fig. 2 Dispersion parameters of receptance FRF matrices of nonparametric subsystems.

receptance FRF matrices of the nonparametric subsystems are generated instead of generating the realizations of the random mass and stiffness matrices and random modal damping parameters of the nonparametric subsystems. In the second case, the subsystems 2, 3, 4, 6, 7, and 8 are precisely characterized by the nonparametric model, unlike the first case. The mean matrices of $\mathbf{H}^{(j)}(\omega)$ ($j = 2, 3, 4, 6, 7, 8$) in the second case are considered to be the same as the mean matrices of the FRF matrices (calculated based on the mean mass and stiffness matrices and mean modal damping parameter considered in case 1) in the first case. The characteristics of the random parameters of the parametric subsystems in the second case are considered to be the same as the respective characteristics in the first case. Again, a total of 575 realizations of the built-up system are generated to use in the MCS technique in the second case.

Based on 575 realizations of the receptance FRF matrices of each uncoupled subsystem in the frequency band of 1–300 Hz for each case, a total of 575 realizations of the responses of the built-up structure for each case is computed by using the coupling technique, as described earlier.

Use of these 575 realizations of the response of the built-up system yields Figs. 3 and 4, showing the statistics of the response $|W_{3,1}|$ of the built-up system, respectively, for case 1 and case 2. In these figures, $|\cdot|$ represents the magnitude of the response and $W_{m,k}$ represents the displacement or rotation denoted by m ($m = 1, \ldots, 6$) (according to the convention described in Fig. 1) at the coupling

Table 1 Uniform distributions for y coordinates of coupling points

Coupling points	y coordinates, m	Subsystem no.
cp_1	$U(0.495, 0.505)$	1
cp_3	$U(0.795, 0.805)$	1
cp_5	$U(1.095, 1.105)$	1
cp_2	$U(0.495, 0.505)$	5
cp_4	$U(0.795, 0.805)$	5
cp_6	$U(1.095, 1.105)$	5
cp_7	$U(0.495, 0.505)$	9
cp_8	$U(0.795, 0.805)$	9
cp_9	$U(1.095, 1.105)$	9

Table 2 Uniform distributions for y coordinates of excitation points

Excitation points	y coordinates, m	Subsystem no.
et_1	$U(1.895, 1.905)$	1
et_2	$U(0.165, 0.175)$	5
et_3	$U(1.895, 1.905)$	5
et_5	$U(1.895, 1.905)$	9

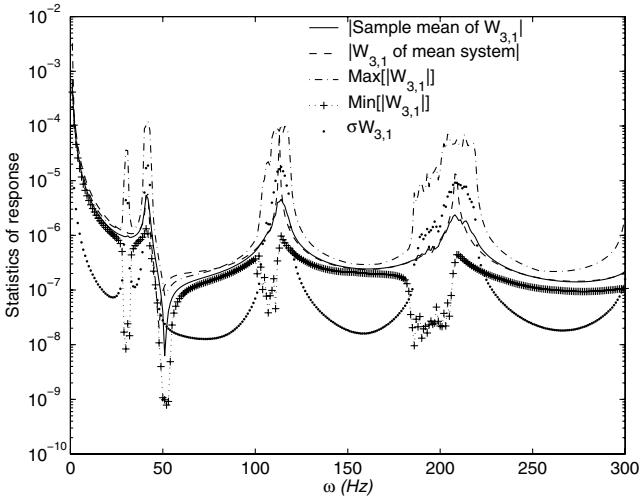

Fig. 3 Statistical details of the deflection $W_{3,1}$ of the built-up structure (case 1).

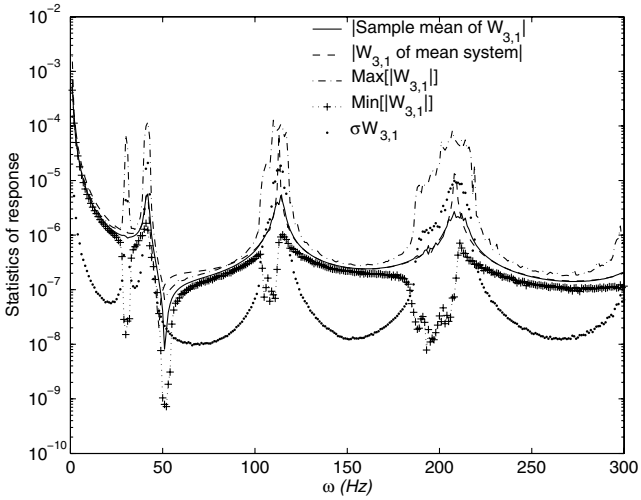

Fig. 4 Statistical details of the deflection, $W_{3,1}$, of the built-up structure (case 2).

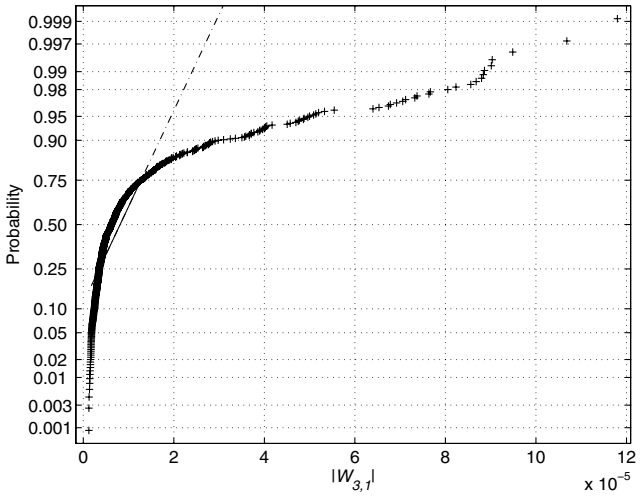

Fig. 6 Normal probability plot of $|W_{3,1}|$ at $\omega = 42$ Hz (case 2).

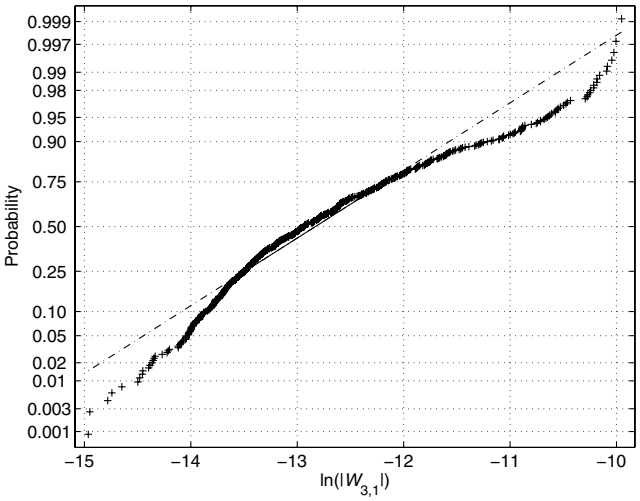

Fig. 7 Normal probability plot of $\ell_n(|W_{3,1}|)$ at $\omega = 208$ Hz (case 1).

point k ($k = 1, \ldots, 9$). It can be seen that all of the statistics (sample mean, sample standard deviation, sample maximum, and sample minimum) of the displacement along the z direction at coupling point 1 computed in both cases match very closely. In these figures, the response of the mean built-up system is also superimposed. Note that the response (same for both cases) of the mean system usually lies in the interval bounded by the sample maximum and sample minimum, except in the frequency range 50–75 Hz near antiresonance for both cases. The responses of the built-up system near a few resonance frequencies of the mean system (i.e., at 42, 113, and 208 Hz) have been separately investigated to see if they follow some of the usual distributions, especially normal or log-normal distribution. It was found (see Figs. 5–8; not all plots are included here) that the response does not follow either distribution. Comparisons of the respective plots of the two cases, however, show that the patterns of the simulated response in both cases are of a similar nature. This also numerically validates the correctness of the nonparametric formulation of the complex symmetric receptance FRF matrix as presented here.

VI. Conclusions

A FRF-based coupling technique to combine the parametric and nonparametric models of stochastic systems has been described. This technique enables the analysis of a complex dynamic systems having spatially nonhomogeneous uncertainty. A complex dynamic system with spatially nonhomogeneous uncertainty can be decomposed into several smaller components such that each component separately shows spatially homogeneous uncertainty over its domain. Consequently, each smaller subsystem is analyzed by using an approach most pertinent to it, and then the results at the subsystem level are assembled by using the coupling technique described here

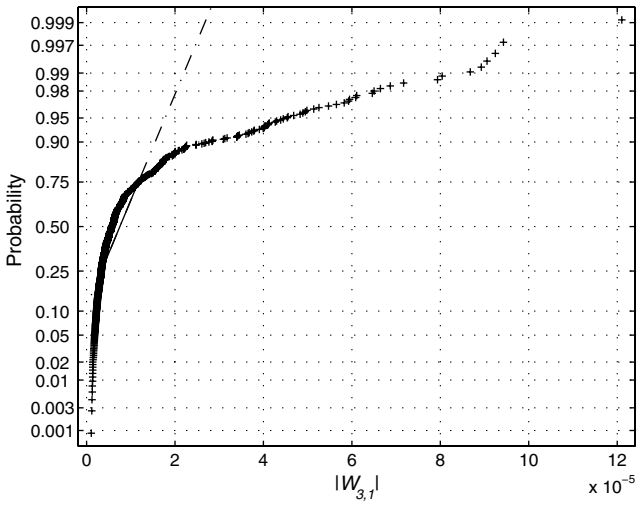

Fig. 5 Normal probability plot of $|W_{3,1}|$ at $\omega = 42$ Hz (case 1).

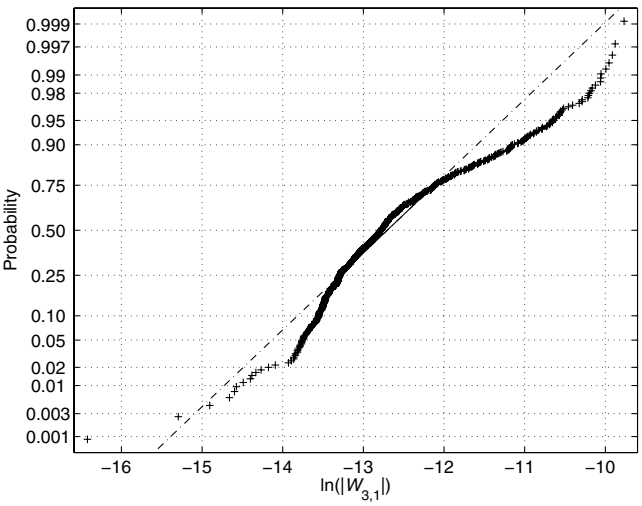

Fig. 8 Normal probability plot of $\ell_n(|W_{3,1}|)$ at $\omega = 208$ Hz (case 2).

to obtain the response quantities of interest at the built-up system level. The FRFs associated with the uncoupled subsystems as required in this coupling technique can be determined analytically or numerically (e.g., by using FEM) as well as experimentally (e.g., through laboratory tests).

Not only the usual real-valued positive-definite/semidefinite mass, stiffness, and damping matrices can be modeled within the framework of the nonparametric approach: it is also shown here that the nonparametric formulation can also be employed to model the uncertainty in the complex-valued symmetric FRF matrix of the system. More generally, even if any system matrix does not show any symmetry (e.g., rotating systems having a skew-symmetric damping matrix or/and skew-symmetric stiffness matrix), it can be effectively dealt with by using the nonparametric approach by having recourse to the SVD that exists for any matrix $A \in \mathbb{M}_{m,n}(\mathbb{C})$.

Acknowledgments

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